Theory of Doppler Radar systems
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1. Basic configuration of Radar system
Microwave radar systems supply a service of detecting the moving objects and of these distances with phenomena of Doppler effects. It is shown a basic configuration of the Doppler Radar systems at the Fig. 1.

![Diagram of Doppler Radar system](image)

These equations of Doppler effects are noted below.

- Transmission radio wave signal at ANT
  \[ y_{tx}(t) = A_0 \cdot \sin(2\pi f_0 t + \phi_0) \]  
  eq. 1

- Received radio wave signal at ANT
  \[ y_{rx}(t) = A_r(t) \cdot \sin(2\pi f_0 t + \phi_L + \phi_0 + \phi_x) \]
  \[ = A_r(t) \cdot \sin\left(2\pi f_0 t - 2\pi \frac{L}{\lambda} + \phi_0 + \phi_r\right) \]
  \[ = A_r(t) \cdot \sin\left(2\pi f_0 t - 2\pi \frac{2R(t)}{\lambda} + \phi_0 + \phi_r\right) \]
  \[ = A_r(t) \cdot \sin\left(2\pi f_0 t - 4\pi \frac{R_0}{\lambda} \cdot \left(R_0 - \nu \cdot t\right) + \phi_0 + \phi_r\right) \]
  \[ = A_r(t) \cdot \sin\left(2\pi f_0 t - \frac{4\pi R_0}{c} \cdot \left(R_0 - \nu \cdot t\right) + \phi_0 + \phi_r\right) \]
  \[ = A_r(t) \cdot \sin\left(2\pi \left(f_0 + f_d\right) t - \frac{4\pi R_0}{\lambda} + \phi_0 + \phi_r\right) \]
  eq. 2

Here,
- \( f_0 [\text{Hz}] \): transmission carrier frequency
- \( \phi_0 [\text{rad}] \): initial phase of transmission wave
- \( \phi_r [\text{rad}] \): phase shift via target object
- \( \phi_L [\text{rad}] \): phase shift through path length \( L = 2R(t) \)
- \( \lambda [\text{m}] \): wavelength of carrier wave
- \( \nu [\text{m/s}] \): velocity of target object
$u > 0$ is getting closer, $u < 0$ is getting away

$L [m]$ : Round-trip distance

$R (t) [m]$ : distance between Radar ANT and target object

$fd [Hz]$ : doppler frequency

$fd > 0$ is getting closer, $fd < 0$ is getting away

c [$m/s$] : velocity of light. $c = f_0 \cdot \lambda$

Here, the Doppler frequency is

$$f_d = \frac{2f_0 \cdot v}{c}$$

eq. 3

The velocity of target object is

$$v = \frac{f_d \cdot c}{2f_0} = \frac{f_d \cdot \lambda}{2}$$

or

$$v = \frac{f_d \cdot c}{2f_0 \cdot \cos \alpha} = \frac{f_d \cdot \lambda}{2 \cdot \cos \alpha}$$

eq. 4

If the direction of moving object has angle $\alpha$ to Radar ANT, doppler frequency $f_d$ is

$$f_d = \frac{2f_0 \cdot v \cdot \cos \alpha}{c}$$

eq. 5

2. Equation of Radar

The transmission power density via isotropic ANT at distance $R [m]$ is

$$\frac{P_{tx}}{4\pi \cdot R^2} \left[ \frac{W}{m^2} \right]$$

eq. 6

Here, $P_{tx} [W]$ is transmission power of radar.

The transmission power density via directional ANT with gain $G_t [(W/\text{rad}^2)/(W/\text{rad}^2)]$ at distance $R [m]$ is

$$\frac{P_{tx} \cdot G_t}{4\pi \cdot R^2} \left[ \frac{W}{m^2} \right]$$

eq. 7

The reflection power via target with reflection cross section $\sigma_0 [m^2]$ is

$$\frac{P_{tx} \cdot G_t \cdot \sigma_0}{4\pi \cdot R^2} [W]$$

eq. 8
The received power density at received location through round-trip path is

\[
\frac{P_\text{in} \cdot G_t \cdot \sigma_0}{(4\pi \cdot R^2)^2 \cdot m^2}
\]

eq. 9

The received power at received ANT with effective area \(A_r [m^2]\) is

\[
\frac{P_\text{in} \cdot G_t \cdot \sigma_0 \cdot A_r}{(4\pi \cdot R^2)^2} [W]
\]

eq. 10

The relation between effective area \(A_r\) of received ANT and gain \(G_r [W/(rad^2)]\) of received ANT is

\[
A_r [m^2] = \frac{G_r \cdot \lambda^2}{4\pi}
\]

eq. 11

Substitute eq. 11 into eq. 10, then the received power \(P_{rx} [W]\) at received ANT is

\[
P_{rx} [W] = \frac{P_\text{in} \cdot G_t \cdot G_r \cdot \sigma_0 \cdot \lambda^2}{(4\pi)^3 \cdot R^4}
\]

eq. 12

Assumed that minimum received power of receiver \(P_{rx}\) is equal to \(S_{min} [W]\), then the maximum detectable range \(R_{max}\) is derived from eq. 12.

\[
R_{max} [m] = \left( \frac{P_\text{in} \cdot G_t \cdot G_r \cdot \sigma_0 \cdot \lambda^2}{(4\pi)^3 \cdot S_{min}} \right)^{1/4}
\]

eq. 13

Power of noise is

\[
S_{min} [W] = k \sqrt{W \cdot \sec \cdot F \cdot T[K] \cdot \Delta BW[Hz] \cdot F}
\]

eq. 14

Noise figure is

\[
F_{total} = F_1 + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1 \cdot G_2} + \Lambda
\]

eq. 15
3. Principle of mixing function with square-law detection in diode's non-linearity

The mixing function for two microwaves apply a diode's non-linearity called square-law detection. The equation of diode characteristics is

\[ i_d[A] = i_s[A] \times \exp \left( \frac{V_d}{V_T} - 1 \right) \]

**eq. 16**

Here,
- \( i_d[A] \) : diode current
- \( i_s[A] \) : diode's reverse bias saturation current. Nearly equal to 1uA for schottky diode.
- \( V_d \) [V]: diode voltage
- \( V_T \) [V]: thermal voltage. \( V_T = kT/q = 26 \text{mV} \) at +25deg C.

![Diode characteristics eq. 16](image)

Fig. 2 Diode characteristics eq. 16

The turning-on voltage of \( V_d \) for schottky diode is nearly equal to 0.3V. So \( V_d/V_T \approx 11.5 >> 1 \).

The eq. 16 is approximated by the following equation

\[ i_d[A] \approx i_s[A] \times \exp \left( \frac{V_d}{V_T} \right) \]

**eq. 17**

Expand the eq. 17 into taylor series.

\[ i_d = i_s \times \left\{ 1 + \frac{V_d}{V_T} + \frac{1}{2!} \left( \frac{V_d}{V_T} \right)^2 + \frac{1}{3!} \left( \frac{V_d}{V_T} \right)^3 + \Lambda \right\} \]

**eq. 18**
Heterodyne or Homodyne detection system apply the diode as mixer with its non-linearity at eq. 18. We describe the case of Homodyne detection. The Homodyne detection of Doppler radar system apply a transmission wave as local oscillator and a reflected signal as received wave.

![Diagram of Doppler radar system](image)

**Fig. 3 The Homodyne detection of Doppler radar system**

The diode voltage $V_d$ is expressed such as eq. 19 from eq. 1, eq. 2 at configuration Fig. 3 according to the principle of superposition.

$$V_d = y_{tx}(t) + y_{rx}(t)$$

**eq. 19**

At eq. 18, a mixing effect is derived only from square term, that is, term of square-law detection.

$$i_d = \frac{i_s}{2V_T^2} \cdot (V_d)^2 = \frac{i_s}{2V_T^2} \cdot (y_{tx}(t) + y_{rx}(t))^2$$

$$= \frac{i_s}{2V_T^2} \cdot \left\{ (y_{tx}(t))^2 + (y_{rx}(t))^2 + 2 \cdot y_{tx}(t) \cdot y_{rx}(t) \right\}$$

**eq. 20**

Another high frequency terms are eliminated by smoothing capacitor at diode detector. Also the terms of $(y_{tx})^2$ and of $(y_{tx})^2$ at eq. 20 are eliminated by smoothing capacitor cause of being high frequency terms.

As a result, the detected diode current with square-law detection through diode mixer is

$$i_{d, \text{Doppler}}(t) = \frac{i_s}{V_T} \cdot y_{tx}(t) \cdot y_{rx}(t)$$

**eq. 21**
4. Single phase Doppler Measurements Systems

Single phase Doppler Measurements Systems is Fig. 3 self. We substitute eq. 1 and eq. 2 into eq. 21. Then, we get a I-signal eq. 22 with smoothing

\[ i_{d,\text{Doppler}}(t) = \frac{i_{r}}{V_{r}} \cdot y_{rs}(t) \cdot y_{rs}(t) \]

\[ = \frac{i_{r}}{V_{r}} \cdot A_{0} \cdot \sin(2\pi f_{d}t + \varphi_{0}) \cdot A_{i}(t) \cdot \sin \left(2\pi(f_{0} + f_{d})t - \frac{4\pi R_{0}}{\lambda} + \varphi_{0} + \varphi_{r} \right) \]

\[ = A_{0}' A_{i}(t) \cdot \sin(2\pi f_{0}t + \varphi_{0}) \cdot \sin \left(2\pi(f_{0} + f_{d})t - \frac{4\pi R_{0}}{\lambda} + \varphi_{0} + \varphi_{r} \right) \]

\[ = \frac{A_{0}' A_{i}(t)}{2} \cdot \left\{ \cos \left(2\pi f_{d}t - \frac{4\pi R_{0}}{\lambda} + \varphi_{r} \right) - \cos \left(2\pi(2f_{0} + f_{d})t - \frac{4\pi R_{0}}{\lambda} + 2\varphi_{0} + \varphi_{r} \right) \right\} \]

\[ \Rightarrow \frac{A_{0}' A_{i}(t)}{2} \cdot \cos \left(2\pi f_{d}t - \frac{4\pi R_{0}}{\lambda} + \varphi_{r} \right) \]

eq. 22

5. Two phase Doppler Measurements Systems

Two phase Doppler Measurements Systems is Fig. 4 self. We get a I-signal as eq. 22. A Q-signal is derived from eq. 22 with adding a phase shift \( \pi/2 \) into receiver signal term.

![Two phase Doppler Measurements Systems](image)

A I-signal voltage is

\[ V_{IF,I}^{\text{Doppler}}[V] = \frac{R_{l}[\Omega]}{2} \cdot \frac{A_{0}' A_{i}(t)}{2} \cdot \cos \left(2\pi f_{d}t - \frac{4\pi R_{0}}{\lambda} + \varphi_{r} \right) \]

eq. 23
A Q-signal voltage is
\[
V_{\text{Doppler}}^{\text{W}} [V] = \frac{R_L [\Omega] \cdot A'_0 A_s (t)}{2} \cdot \cos \left( 2\pi f_d t - \frac{4\pi R_0}{\lambda} + \varphi_r - \frac{\pi}{2} \right)
\]
\[
= \frac{R_L [\Omega] \cdot A'_0 A_s (t)}{2} \cdot \sin \left( 2\pi f_d t - \frac{4\pi R_0}{\lambda} + \varphi_r \right)
\]
\text{eq. 24}

Here, RL[Ω] is load resistor at schottky diode. A0’ is
\[
A'_0 \left[ \frac{A}{V} \right] = \frac{i_s}{V_t^2} \cdot A_0 [V]
\]
\text{eq. 25}

We are going to calculate the amplitude and phase of received signal from eq.: 23, eq.: 24.

Amplitude = \[
\sqrt{\left( V_{\text{Doppler}}^{\text{W}, \text{I}} \right)^2 + \left( V_{\text{Doppler}}^{\text{W}, \text{Q}} \right)^2}
\]
\[
= \frac{R_L [\Omega] \cdot A'_0 A_s (t)}{2} \times \sqrt{\left( \cos \left( 2\pi f_d t - \frac{4\pi R_0}{\lambda} + \varphi_r \right) \right)^2 + \left( \sin \left( 2\pi f_d t - \frac{4\pi R_0}{\lambda} + \varphi_r \right) \right)^2}
\]
\[
= \frac{R_L [\Omega] \cdot A'_0 A_s (t)}{2}
\]
\text{eq. 26}

Phase(t) = \theta(t) = \tan^{-1} \left( \frac{V_{\text{Doppler}}^{\text{W}, \text{Q}}}{V_{\text{Doppler}}^{\text{W}, \text{I}}} \right)
\text{eq. 27}

\[
= \tan^{-1} \left( \frac{\sin \left( 2\pi f_d t - \frac{4\pi R_0}{\lambda} + \varphi_r \right)}{\cos \left( 2\pi f_d t - \frac{4\pi R_0}{\lambda} + \varphi_r \right)} \right)
\]
\[
= \tan^{-1} \left( \tan \left( 2\pi f_d t - \frac{4\pi R_0}{\lambda} + \varphi_r \right) \right)
\]
\[
= 2\pi f_d t - \frac{4\pi R_0}{\lambda} + \varphi_r
\]
\text{eq. 28}

We can calculate the Doppler frequency \( f_d \) from the derivation of phase \( \theta(t) \).

\[
f_d = \frac{1}{2\pi} \cdot \frac{d}{dt} \theta(t) = \frac{1}{2\pi} \cdot \frac{\theta(t_2) - \theta(t_1)}{t_2 - t_1}
\]
\text{eq. 28}
5.1 Approaching and moving away of the moving objects

It is convenient for us to apply the Doppler frequency \( f_d > 0 \) as for approaching, and \( -f_d < 0 \) as for moving away. We replace eq. 23, eq. 24 into from eq. 29 to eq. 32. Here we omit the constant phase terms from equation.

In the case of moving away

\[
V_{IF-I}^Doppler[V] \Rightarrow f(t)_{moving\_away} = I \cdot \cos(2\pi f_d t) \quad eq. \ 29
\]

\[
V_{IF-Q}^Doppler[V] \Rightarrow g(t)_{moving\_away} = Q \cdot \cos(-2\pi f_d t - \frac{\pi}{2}) = Q \cdot \cos(2\pi f_d t + \frac{\pi}{2}) \quad eq. \ 30
\]

In the case of approaching

\[
V_{IF-I}^Doppler[V] \Rightarrow f(t)_{approaching} = I \cdot \cos(-2\pi f_d t) = I \cdot \cos(2\pi f_d t) \quad eq. \ 31
\]

\[
V_{IF-Q}^Doppler[V] \Rightarrow g(t)_{approaching} = Q \cdot \cos(2\pi f_d t - \frac{\pi}{2}) \quad eq. \ 32
\]

We can always get the \( \pm \frac{\pi}{2} \) phase shift for Q-signals to I-signals. See Fig. 5.

![Fig. 5 ± π/2 phase shift for Q-signals to I-signals](image)

5.2 Complex Fourier Transform for I,Q-signals and processing of approaching and of moving away

We can apply the Fourier Transform to I-signal \( f(t) \) and to Q-signal \( g(t) \) which have the distribution of some Doppler frequencies. For I-signal \( f(t) \), we apply complex Fourier Transform

\[
f(t) = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} F(\omega) \cdot e^{i\omega t} d\omega = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} |F(\omega)| \cdot \exp(j \cdot (\omega \cdot t + \theta)) d\omega \quad eq. \ 33
\]
Here,
\[ F(\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t} dt = a(\omega) + j \cdot b(\omega) = |F(\omega)| \cdot \exp(j \cdot \theta_{i}) \] \hspace{1cm} \text{eq. 34}

\[ a(\omega) = \text{Re}(F(\omega)), b(\omega) = \text{Im}(F(\omega)), |F(\omega)| = \sqrt{a^{2} + b^{2}}, \theta_{i} = \tan^{-1}\left(\frac{b(\omega)}{a(\omega)}\right) \]

For Q-signal \( g(t) \),
\[ g(t) = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} G(\omega) \cdot e^{-j\omega t} d\omega = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} |G(\omega)| \cdot \exp(j \cdot (\omega \cdot t + \theta_{\omega})) d\omega \] \hspace{1cm} \text{eq. 35}

Here,
\[ G(\omega) = \int_{-\infty}^{\infty} g(t) \cdot e^{-j\omega t} dt = a'(\omega) + j \cdot b'(\omega) = |G(\omega)| \cdot \exp(j \cdot \theta_{\omega}) \] \hspace{1cm} \text{eq. 36}

\[ a'(\omega) = \text{Re}(G(\omega)) \cdot b'(\omega) = \text{Im}(G(\omega)) \cdot |G(\omega)| = \sqrt{a'^{2} + b'^{2}}, \theta_{i} = \tan^{-1}\left(\frac{b'(\omega)}{a'(\omega)}\right) \]

Then we get the fraction of complex Fourier Transform for each I,Q signals.

\[ \frac{G(\omega)}{F(\omega)} = \frac{a'(\omega) + j \cdot b'(\omega)}{a(\omega) + j \cdot b(\omega)} = \frac{a(\omega) \cdot a'(\omega) + b(\omega) \cdot b'(\omega)}{a(\omega)^{2} + b(\omega)^{2}} + j \cdot \frac{a(\omega) \cdot b'(\omega) - b(\omega) \cdot a'(\omega)}{a(\omega)^{2} + b(\omega)^{2}} \]

\[ = \frac{|G(\omega)| \cdot \exp(j \cdot \theta_{\omega})}{|F(\omega)| \cdot \exp(j \cdot \theta_{i})} = \frac{|G(\omega)|}{|F(\omega)|} \cdot \exp(j \cdot (\theta_{\omega} - \theta_{i})) \] \hspace{1cm} \text{eq. 37}

Now we can assume approximations such as below.

\[ |F(\omega)| \approx |G(\omega)| \]

\[ \text{eq. 37 becomes to eq. 38} \]

\[ \exp(j \cdot (\theta_{\omega} - \theta_{i})) = \frac{a(\omega) \cdot a'(\omega) + b(\omega) \cdot b'(\omega)}{a(\omega)^{2} + b(\omega)^{2}} + j \cdot \frac{a(\omega) \cdot b'(\omega) - b(\omega) \cdot a'(\omega)}{a(\omega)^{2} + b(\omega)^{2}} \] \hspace{1cm} \text{eq. 38}

Then,
\[ \tan(\theta_{\omega} - \theta_{i}) = \frac{a(\omega) \cdot b'(\omega) - b(\omega) \cdot a'(\omega)}{a(\omega)^{2} + b(\omega)^{2}} \cdot \frac{a(\omega)^{2} + b(\omega)^{2}}{a(\omega) \cdot a'(\omega) + b(\omega) \cdot b'(\omega)} \]
\[ = \frac{a(\omega) \cdot b'(\omega) - b(\omega) \cdot a'(\omega)}{a(\omega) \cdot a'(\omega) + b(\omega) \cdot b'(\omega)} \approx \tan\left(\pm \frac{\pi}{2}\right) \] \hspace{1cm} \text{eq. 39}

So we are able to check the direction of approaching and of moving away via eq. 39.

By the way, we can get a time-waveform function of the actual real signal.
\[ f(t) = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} \sqrt{a(\omega)^{2} + b(\omega)^{2}} \cdot \cos(\omega \cdot t + \theta_{i}(\omega)) d\omega \] \hspace{1cm} \text{eq. 40}
6. Principle of FM-CW Doppler Radar measurement

The FM-CW Doppler Radar measurement enables us to measure a distance to the object with Linear (Ramp) Frequency Modulation for carrier wave such as 

\[ y_{rx}(t) = A_0 \cdot \sin(2\pi f_0(t) \cdot t + \varphi_0) = A_0 \cdot \sin(2\pi(f_0 \pm \alpha t) \cdot t + \varphi_0) \]

\[ \text{eq.: 41} \]

\[ y_{rx}(t) = A_r(t) \cdot \sin\left(2\pi\left(f_0(t) + f_d\right)t - \frac{4\pi R_0}{\lambda} + \varphi_0 + \varphi_r\right) \]

\[ = A_r(t) \cdot \sin\left(2\pi\left(f_0 + \Delta f\right) + f_d\right)t - \frac{4\pi R_0}{\lambda} + \varphi_0 + \varphi_r\right) \]

\[ = A_r(t) \cdot \sin\left(2\pi\left(f_0 \pm \alpha \left(t - \frac{2R(t)}{c}\right) + f_d\right) - \frac{4\pi R_0}{\lambda} + \varphi_0 + \varphi_r \right) \]

\[ \text{eq.: 42} \]

Here, \( f_w \) is sweep bandwidth of carrier FM, \( t_{swp} \) is sweep time, \( \alpha \) is 

\[ \alpha [\text{Hz/sec}] = \frac{f_w}{t_{swp}} \]

\[ \text{eq.: 43} \]

**Fig. 6** FM-CW method
We substitute eq. 42 and eq. 43 into eq. 21. Then, we get a I-signal eq. 44 with smoothing
\[ i_d^{FM-CW}(t) = \frac{i_r}{V_r^2} \cdot y_{\alpha}(t) \cdot y_{\tau}(t) \]
\[ = \frac{i_r}{V_r^2} \cdot A_0 \cdot \sin(2\pi f_0(t) \cdot t + \phi_0) \cdot A_r(t) \cdot \sin \left(2\pi \left(f_0(t) + f_d\right) \cdot t - \frac{4\pi R_0}{\lambda} + \phi_0 + \phi_r\right) \]
\[ = A' \cdot A_r(t) \cdot \sin(2\pi \left(f_0 \pm \alpha \cdot t + \phi_0\right) \cdot \sin \left(2\pi \left(f_0 \pm \alpha \cdot \left(t - \frac{2R(t)}{c}\right) + f_d\right) - \frac{4\pi R_0}{\lambda} + \phi_0 + \phi_r\right) \]
\[ = A' \cdot A_r(t) \cdot \left\{ \cos \left(2\pi f_0 \cdot t - \frac{4\pi R_0}{\lambda} + \phi_r \right) + 2\alpha \cdot 2R(t) \cdot t \right\} \]
\[ \Rightarrow A' \cdot A_r(t) \cdot \cos \left(2\pi f_0 \cdot t - \frac{4\pi R_0}{\lambda} + \phi_r \right) \]
\[ = \frac{A' \cdot A_r(t)}{2} \cdot \cos \left(2\pi f_d \cdot t - \frac{4\pi R_0}{\lambda} + \phi_r \right) \]
\[ = \frac{A' \cdot A_r(t)}{2} \cdot \cos \left(2\pi f_d \cdot t - \frac{4\pi R_0}{\lambda} + \phi_r \right) \]
\[ = \frac{A' \cdot A_r(t)}{2} \cdot \cos \left(2\pi f_d \cdot t - \frac{4\pi R_0}{\lambda} + \phi_r \right) \]
\[ \text{eq. 44} \]

Here, the FM-CW beat frequency is
\[ f_{beat}^{FM-CW}[Hz] = \alpha \cdot \frac{2R(t)}{c} \]
\[ \text{eq. 45} \]

A I-signal voltage in the case of FM-CW measurement is
\[ V_{IF-I}^{FM-CW}[V] = \frac{R_t \left[ \Omega \right]}{2} \cdot A' \cdot A_r(t) \cdot \cos \left(2\pi \left(f_{beat}^{FM-CW} \pm f_d\right) \cdot t - \frac{4\pi R_0}{\lambda} + \phi_r \right) \]
\[ \text{eq. 46} \]

A Q-signal voltage in the case of FM-CW measurement is
\[ V_{IF-Q}^{FM-CW}[V] = \frac{R_t \left[ \Omega \right]}{2} \cdot A' \cdot A_r(t) \cdot \sin \left(2\pi \left(f_{beat}^{FM-CW} \pm f_d\right) \cdot t - \frac{4\pi R_0}{\lambda} + \phi_r \right) \]
\[ \text{eq. 47} \]
We are going to calculate the amplitude and phase of received signal from eq. 46, eq. 47.

Amplitude = \[ \sqrt{\left( V_{IF_{Q}}^{FM-CW} \right)^2 + \left( V_{IF_{I}}^{FM-CW} \right)^2} \]

\[ = \frac{R_{L} [\Omega] \cdot A'_{0} \cdot A_{r}(t)}{2} \times \left\{ \cos \left( 2\pi \left( f_{\text{beat}}^{FM-CW} \pm f_{d} \right) \cdot t - \frac{4\pi R_{0}}{\lambda} + \varphi \right) \right\}^2 + \left\{ \sin \left( 2\pi \left( f_{\text{beat}}^{FM-CW} \pm f_{d} \right) \cdot t - \frac{4\pi R_{0}}{\lambda} + \varphi \right) \right\}^2 \]

\[ = \frac{R_{L} [\Omega] \cdot A'_{0} \cdot A_{r}(t)}{2} \]

\[ \text{eq. 48} \]

Phase(t) = \[ \theta(t) = \tan^{-1} \left( \frac{V_{IF_{Q}}^{FM-CW}}{V_{IF_{I}}^{FM-CW}} \right) \]

\[ = \tan^{-1} \left( \sin \left( 2\pi \left( f_{\text{beat}}^{FM-CW} \pm f_{d} \right) \cdot t - \frac{4\pi R_{0}}{\lambda} + \varphi \right) \right) \cos \left( 2\pi \left( f_{\text{beat}}^{FM-CW} \pm f_{d} \right) \cdot t - \frac{4\pi R_{0}}{\lambda} + \varphi \right) \]

\[ = \tan^{-1} \left( \tan \left( 2\pi \left( f_{\text{beat}}^{FM-CW} \pm f_{d} \right) \cdot t - \frac{4\pi R_{0}}{\lambda} + \varphi \right) \right) \]

\[ = 2\pi \left( f_{\text{beat}}^{FM-CW} \pm f_{d} \right) \cdot t - \frac{4\pi R_{0}}{\lambda} + \varphi \]

\[ \text{eq. 49} \]

We can calculate the Doppler frequency \( f_{d} \) and FM-CW beat frequency \( f_{\text{beat}}^{FM-CW} \) from the derivation of phase \( \theta(t) \).

\[ f_{\text{beat}}^{FM-CW} \pm f_{d} = \frac{1}{2\pi} \cdot \frac{d}{dt} \theta(t) = \frac{1}{2\pi} \cdot \frac{\theta(t_2) - \theta(t_1)}{t_2 - t_1} \]

\[ \text{eq. 50} \]

Then, we are going to solve the relations between a resolution of distance and of beat frequency. From eq. 45, we get the following equations.

\[ R[m] = \frac{c [m/\text{sec}]}{2 \cdot \alpha [Hz/\text{sec}]} \left( f_{\text{beat}}^{FM-CW} \right) [Hz] \]

\[ \text{eq. 51} \]

\[ \Delta R[m] = \frac{c [m/\text{sec}]}{2 \cdot \alpha [Hz/\text{sec}]} \left( \Delta f_{\text{beat}}^{FM-CW} \right) [Hz] \]

\[ \text{eq. 52} \]
The Doppler module for FM-CW has a VCO (Voltage Controlled Oscillator) with tune Voltage Vt. The VCO has a characteristic as to a slope with \( \beta [\text{MHz/V}] \) such as below.

\[
\Delta f_{\text{beat}}\left[\text{Hz}\right] = \alpha\left[\text{Hz/}\text{sec}\right] \cdot \frac{2 \cdot \Delta R [\text{m}]}{c [\text{m/}\text{sec}]} \quad \text{eq. \# 53}
\]

That is, at eq. \# 42, we get

\[
\beta\left[\text{MHz/}V\right] = \frac{df_{\text{vco}} [\text{MHz}]}{dVt[V]} \quad \text{eq. \# 54}
\]

So, from eq. \# 53 and eq. \# 54

\[
\Delta f_{\text{vco}} [\text{MHz}] = \Delta f [\text{Hz}] = \alpha\left[\text{Hz/}\text{sec}\right] \cdot \Delta t [\text{sec}] = \Delta f_{\text{beat}} \left[\text{Hz}\right] \quad \text{eq. \# 55}
\]

Then,

\[
\Delta f_{\text{vco}} [\text{MHz}] = \beta\left[\text{MHz/}V\right] \cdot \Delta Vt[V] = \alpha\left[\text{Hz/}\text{sec}\right] \cdot \frac{2 \cdot \Delta R [\text{m}]}{c [\text{m/}\text{sec}]} \quad \text{eq. \# 56}
\]

or

\[
\Delta Vt[V] = \frac{\alpha\left[\text{Hz/}\text{sec}\right]}{\beta\left[\text{MHz/}V\right]} \cdot \frac{2 \cdot \Delta R [\text{m}]}{c [\text{m/}\text{sec}]} \quad \text{eq. \# 57}
\]

\[
\Delta R [\text{m}] = \frac{\beta\left[\text{MHz/}V\right] \cdot c [\text{m/}\text{sec}]}{2 \cdot \alpha\left[\text{Hz/}\text{sec}\right]} \cdot \Delta Vt[V] \quad \text{eq. \# 58}
\]
7. Principle of FSK Doppler Radar measurement

We assume that carrier frequency \( f_1 \) and \( f_2 \) exist simultaneously, then we get two I-signals of Doppler.

\[
V_{\text{IF}, f_1}^{\text{Doppler}}[V] = \frac{R_t}{2} \cdot A'_0 A_r(t) \cdot \cos \left( 2\pi f_{d1}t - \frac{4\pi R_0}{\lambda_1} + \varphi_r \right)
\]

\( \text{eq.} \ 59 \)

\[
V_{\text{IF}, f_2}^{\text{Doppler}}[V] = \frac{R_t}{2} \cdot A'_0 A_r(t) \cdot \cos \left( 2\pi f_{d2}t - \frac{4\pi R_0}{\lambda_2} + \varphi_r \right)
\]

\( \text{eq.} \ 60 \)

\[
f_{d1} = \frac{2f_1 \cdot v}{c} \cdot \cos \alpha
\]

\( \text{eq.} \ 61 \)

\[
f_{d2} = \frac{2f_2 \cdot v}{c} \cdot \cos \alpha
\]

\( \text{eq.} \ 62 \)

**Fig. 7** Two Doppler signals for \( f_1 \) and \( f_2 \)

We compare two Doppler I-signals for \( f_1 \) and \( f_2 \) with the condition of same amplitude. Then

\[
V_{\text{IF}, f_1}^{\text{Doppler}}(t = t_1) = V_{\text{IF}, f_2}^{\text{Doppler}}(t = t_2)
\]

\[
\cos \left( 2\pi f_{d1}t_1 - \frac{4\pi R_0}{\lambda_1} + \varphi_r \right) = \cos \left( 2\pi f_{d2}t_2 - \frac{4\pi R_0}{\lambda_2} + \varphi_r \right)
\]

\( \text{eq.} \ 63 \)
The phase is same at both sides of eq. 63. Then

\[ 2\pi f_{d1} t_1 - \frac{4\pi R_0}{\lambda_1} = 2\pi f_{d2} t_2 - \frac{4\pi R_0}{\lambda_2} \]

\[ \frac{4\pi R_0}{\lambda_2} - \frac{4\pi R_0}{\lambda_1} = 2\pi f_{d2} t_2 - 2\pi f_{d1} t_1 \]

\[ 4\pi \left( \frac{1}{\lambda_2} - \frac{1}{\lambda_1} \right) \cdot R_0 = \theta_2(t_2) - \theta_1(t_1) \]

\[ \frac{4\pi}{c} (f_2 - f_1) \cdot R_0 = \theta_2(t_2) - \theta_1(t_1) \]

\[ R_0 = \frac{c}{4\pi} \cdot \frac{\Delta \theta}{(f_2 - f_1)} \]

eq. 64

We are able to measure the distance to object by calculating eq. 64.

Actually it is difficult to measure two Doppler signals of \( f_1 \) and \( f_2 \) simultaneously, so FSK (frequency shift keying) is applied to the Doppler measurements for distance such as Fig. 8.

Fig. 8 FSK Doppler measurement
8. Signal processing for Radar systems

Signal processing for Radar systems apply FFT (Fast Fourier Transform) algorithm for detecting the amplitude and the phase of Doppler signals in a lot of cases. We sum up some formula about Fourier Transform. General Signal function \( f(t) \) is expressed via Fourier Transform \( F(\omega) \).

\[
f(t) = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} F(\omega) \cdot e^{j\omega t} \, d\omega
\]

eq. 65

\[
F(\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t} \, dt
\]

eq. 66

9. DFT (Discrete Fourier Transform)

Actually we apply a discrete time-sampling via ADC (Analog to Digital Converter), so continuous signal \( f(t) \) is converted into discrete sampling signal \( f_s(t) \).

\[
f_s(t) = f(t) \cdot \delta_T(t) = \sum_{n=-\infty}^{\infty} \{f(nT) \times \delta(t - nT)\}
\]

eq. 67

The Fourier Transform \( F_s(\omega) \) of discrete sampling signal \( f_s(t) \) is

\[
f_s(t) \leftrightarrow F_s(\omega) = \frac{\omega_{\text{sample}}}{2\pi} \left[ F(\omega) \ast \delta_{\omega_{\text{sample}}} (\omega) \right] = \frac{1}{T} \times \left[ F(\omega) \ast \delta_{\omega_{\text{sample}}} (\omega) \right] = \frac{1}{T} \times \sum_{n=-\infty}^{\infty} F(\omega - n\omega_{\text{sample}})
\]

eq. 68

Here, \( \omega_{\text{sample}}[\text{Hz}] = 1/T[\text{sec}] \), \( \omega_{\text{sample}}[\text{rad}] = 2\pi \times \omega_{\text{sample}} \), \( \delta \) is delta function, \( \delta_T(t) \), \( \delta_{\omega_{\text{sample}}} (\omega) \) is impulse train at each time region and angular frequency region

\[
\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)
\]

eq. 69

\[
\delta_{\omega_{\text{sample}}} (\omega) = \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_{\text{sample}})
\]

eq. 70

\* is a Convolution Integral

\[
F(\omega) \ast \delta_{\omega_{\text{sample}}} (\omega) = \int_{-\infty}^{\infty} F(u) \cdot \left( \sum_{n=-\infty}^{\infty} \delta(\omega - u - n\omega_{\text{sample}}) \right) \, du
\]

\[
= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} F(u) \cdot \delta(\omega - u - n\omega_{\text{sample}}) \, du
\]

\[
= \sum_{n=-\infty}^{\infty} F(\omega - n\omega_{\text{sample}})
\]

eq. 71
The original time waveform signal \( f(t) \) is restored with having restricted the bandwidth of \( F_s(\omega) \) via rectangular form Low Pass Filter \( \pi/\sigma \times P_\sigma(\omega) \), \( T=\pi/\sigma \), so we can calculate the convolution integral of \( F_s(\omega) \times \pi/\sigma \times P_\sigma(\omega) \), then we get the original signal \( f(t) \)

\[
f(t) = f_s(t) * S_\sigma(\sigma) \\
= \sum_{m=-\infty}^{\infty} f(nT) \delta(t - nT) * S_\sigma(\sigma) \\
= \sum_{m=-\infty}^{\infty} f(nT) \cdot S_\sigma(\sigma(t - nT))
\]

eq.: 72

Here, \( S_\sigma(\sigma) \) is a sampling function

\[
S_\sigma(\sigma) = \frac{\sin(\sigma)}{\sigma}
\]

eq.: 73

In a similar way, we get original spectrum signal \( F(\omega) \)

\[
F(\omega) = \sum_{n=-\infty}^{\infty} F(n\Omega) \cdot S_\sigma(\sigma(\omega - n\Omega))
\]

eq.: 74

Here, \( \Omega [\text{rad}]=\pi/\tau \), \( f(t)=0 \) at the condition \( |t|>\tau \).

We need Poisson sum formula in order to acquire the FFT algorithm. The Poisson sum formula is derived such as below.

\[
f(t) \ast \delta_{T_p}(t) = \int_{-\infty}^{\infty} f(\tau) \left( \sum_{m=-\infty}^{\infty} \delta(t - \tau - nT_p) \right) \cdot d\tau \\
= \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau) \cdot \delta(t - \tau - nT_p) \cdot d\tau \\
= \sum_{m=-\infty}^{\infty} f(t - nT_p) = \sum_{m=-\infty}^{\infty} f(t + mT_p)
\]

eq.: 75

Here, \( T_p=NT \), \( T \) is time sampling period for \( f(t) \), \( N \) is arbitrary positive number

Transform eq.: 75 into Fourier form

\[
\text{FourierTransform}\left[f(t) \ast \delta_{T_p}(t)\right] = F(\omega) \times \text{FourierTransform}\left[\delta_{T_p}(t)\right] \\
= F(\omega) \times \text{FourierTransform}\left[\sum_{m=-\infty}^{\infty} \delta(t - nT_p)\right] \\
= F(\omega) \times \Omega \cdot \delta_{\Omega}(\omega) \\
= F(\omega) \times \Omega \cdot \sum_{n=-\infty}^{\infty} \delta(\omega - n\Omega) \\
= \frac{2\pi}{T_p} \cdot \sum_{n=-\infty}^{\infty} F(n\Omega) \cdot \delta(\omega - n\Omega)
\]

eq.: 76
Here, $\Omega=2\pi/T_p=\omega_{\text{sample}}/N=2\pi/NT$. By the way, eq. 75 is periodic function with $T_p$, $n$ is integer number. We replace eq. 75 by new variable parameter.

$$\tilde{f}(t) = f(t) * \delta_{T_p}(t) = \sum_{m=-\infty}^{\infty} f(t + mT_p)$$

eq. 77

The eq. 77 is expressed by Fourier Series.

$$\tilde{f}(t) = \sum_{m=-\infty}^{\infty} \alpha_m \cdot e^{j\Omega t}$$

eq. 78

Transform eq. 78 into Fourier form

$$\text{Transform}[\tilde{f}(t)] = \text{Transform}[f(t) * \delta_{T_p}(t)] = 2\pi \sum_{m=-\infty}^{\infty} \alpha_m \cdot \delta(\omega - m\Omega)$$

eq. 79

Compare eq. 76 and eq. 79

$$\alpha_m = \frac{F(m\Omega)}{T_p}$$

eq. 80

Thus, we get

$$\tilde{f}(t) = \frac{1}{T_p} \sum_{m=-\infty}^{\infty} F(m\Omega) \cdot e^{j\Omega t}$$

eq. 81

The eq. 81 is Poisson sum formula. Then, we are also going to seek the basic theorem of discrete sampling system. We sample the signal eq. 81.

$$\tilde{f}(nT) = \frac{1}{T_p} \sum_{k=0}^{N-1} \sum_{r=-\infty}^{\infty} F((k + rN)\Omega) \cdot \exp(jn(k + rN)\Omega T)$$

eq. 82
Here, \( m = k + rN \), \( 0 < k < N - 1 \), \( r \) is integer number. Substitute \( \Omega = \omega_{\text{sample}}/N = 2\pi/NT \) into eq. 82.

\[
\tilde{f}(nT) = \frac{1}{T_p} \cdot \sum_{k=0}^{N-1} \sum_{r=-\infty}^{\infty} F(k\Omega + r \cdot \omega_{\text{sample}}) \cdot \exp\left( j \cdot \frac{2\pi nk}{N} + nr \right)
\]

\[
= \frac{1}{T_p} \cdot \sum_{k=0}^{N-1} \sum_{r=-\infty}^{\infty} F(k\Omega + r \cdot \omega_{\text{sample}}) \cdot \exp\left( j \cdot \frac{2\pi nk}{N} \right)
\]

\text{eq. 83}

We can express the periodic spectrum function such as below.

\[
\tilde{F}(k\Omega) = \sum_{r=-\infty}^{\infty} F(k\Omega + r \cdot \omega_{\text{sample}})
\]

\text{eq. 84}

Here, we replace new variable \( WN \) called twiddle factor such as below.

\[
W_N = e^{-j\frac{2\pi}{N}}
\]

\text{eq. 85}

Then we get

\[
T \cdot \tilde{f}(nT) = \frac{1}{N} \cdot \sum_{k=0}^{N-1} \tilde{F}(k\Omega) \cdot W_N^{-kn}
\]

\text{eq. 86}

The eq. 86 is one of the basic theorem of discrete sampling system. Next we are also going to seek another basic theorem of discrete sampling system. First we calculate following equation.

\[
T \cdot \sum_{n=0}^{N-1} \tilde{f}(nT) \cdot W_N^{kn}
\]

\text{eq. 87}

Substitute eq. 86 into eq. 87.

\[
T \cdot \sum_{n=0}^{N-1} \tilde{f}(nT) \cdot W_N^{kn} = T \cdot \sum_{n=0}^{N-1} \left[ \frac{1}{NT} \cdot \sum_{m=0}^{N-1} \tilde{F}(m\Omega) \cdot W_N^{-km} \right] \cdot W_N^{kn}
\]

\[
= \sum_{m=0}^{N-1} \tilde{F}(m\Omega) \cdot \frac{1}{N} \cdot \sum_{n=0}^{N-1} W_N^{(k-m)n}
\]

\text{eq. 88}
Here, twiddle factor is
\[
\frac{1}{N} \cdot \sum_{n=0}^{N-1} W_N^{(k-m)n} = \begin{cases} 
1 & (m = k) \\
0 & (m \neq k) 
\end{cases}
\]
eq. 89

Only N is even. (N=2p) Now we get the basic theorem of discrete sampling system.
\[
\tilde{F}(k\Omega) = T \sum_{n=0}^{N-1} \tilde{f}(nT) \cdot W_N^{kn}
\]

We replace new periodic variables such as below.
\[
\tilde{x}(n) = \tilde{f}(nT), \quad \tilde{X}(k) = \frac{\tilde{F}(k\Omega)}{T}
\]
eq. 90

Then we get the Discrete Fourier Series (DFS).
\[
\tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{x}(n) \cdot W_N^{kn}
\]
eq. 91
\[
\tilde{x}(n) = \frac{1}{N} \cdot \sum_{k=0}^{N-1} \tilde{X}(k) \cdot W_N^{-kn}
\]
eq. 92

We clip the primary part of DFS eq. 92 in the basic interval. We get the Discrete Fourier Transform (DFT).
\[
X(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn} \quad 0 \leq k \leq N - 1
\]
eq. 93
\[
x(n) = \frac{1}{N} \cdot \sum_{k=0}^{N-1} X(k) \cdot W_N^{-kn} \quad 0 \leq n \leq N - 1
\]
eq. 94
10. Reverse sequence operation

Now we progress to the reverse sequence operation related to FFT algorithm. We separate eq. 93 into two parts.

$$X(k) = \sum_{n=0}^{N/2-1} x(2n) \cdot W_N^{k \cdot 2n} + \sum_{n=0}^{N/2-1} x(2n+1) \cdot W_N^{k \cdot (2n+1)}$$

eq. 95

By the way we get a relationship as for coefficient twiddle factor WN such as below.

$$W_N^2 = e^{-j \frac{2\pi}{N}} = W_N^{N/2}$$

eq. 96

Substitute eq. 96 into eq. 95. We get

$$X(k) = \sum_{n=0}^{N/2-1} x(2n) \cdot W_N^{k \cdot n} + W_N^{k} \cdot \sum_{n=0}^{N/2-1} x(2n+1) \cdot W_N^{k \cdot n} = G(k) + W_N^k \cdot H(k)$$

eq. 97

Here,

$$G(k) = \sum_{n=0}^{N/2-1} x(2n) \cdot W_N^{k \cdot n} \quad H(k) = \sum_{n=0}^{N/2-1} x(2n+1) \cdot W_N^{k \cdot n}$$

eq. 98

eq. 98 apply reusing for twiddle factor W between G(k) and H(k).

Example) Case of N=8 at eq. 93

$$\begin{pmatrix}
X(0) \\
X(1) \\
X(2) \\
X(3) \\
X(4) \\
X(5) \\
X(6) \\
X(7)
\end{pmatrix} = 
\begin{pmatrix}
W^0 & W^0 & W^0 & W^0 & W^0 & W^0 & W^0 & W^0 \\
W^0 & W^1 & W^2 & W^3 & W^4 & W^5 & W^6 & W^7 \\
W^0 & W^2 & W^4 & W^6 & W^8 & W^{10} & W^{12} & W^{14} \\
W^0 & W^3 & W^6 & W^9 & W^{12} & W^{15} & W^{18} & W^{21} \\
W^0 & W^4 & W^8 & W^{12} & W^{16} & W^{20} & W^{24} & W^{28} \\
W^0 & W^5 & W^{10} & W^{15} & W^{20} & W^{25} & W^{30} & W^{35} \\
W^0 & W^6 & W^{12} & W^{18} & W^{24} & W^{30} & W^{36} & W^{42} \\
W^0 & W^7 & W^{14} & W^{21} & W^{28} & W^{35} & W^{42} & W^{49}
\end{pmatrix} \begin{pmatrix}
x(0) \\
x(1) \\
x(2) \\
x(3) \\
x(4) \\
x(5) \\
x(6) \\
x(7)
\end{pmatrix}$$

eq. 99

Here we replace the scale of sample size N=2^m (m : Positive Integer)

In the case of eq. 99, N=8=111b, m=3. We progress into the following bit reverse sequence operation steps.
a) Display the index \( k \) of eq. 93 as binary number.
b) Flip the bit of binary \( k \) from left to right
c) Bring back the binary flipped \( k \) into Decimal
d) Flip the row of vector \( X(k) \) according to the flipped index \( k \)

\[
\begin{array}{c|c|c|c}
 k(\text{Dec}) & \text{Binary} & \text{Flipped} & k(\text{Flipped Dec}) \\
0 & 000 & 000 & 0 \\
1 & 001 & 100 & 4 \\
2 & 010 & 010 & 2 \\
3 & \Rightarrow 011 & \Rightarrow 110 & \Rightarrow 6 \\
4 & 100 & 001 & 1 \\
5 & 101 & 101 & 5 \\
6 & 110 & 011 & 3 \\
7 & 111 & 111 & 7 \\
\end{array}
\]

Fig. 9 bit reverse sequence operation

For example about eq. 99 with bit reverse sequence operation

\[
\begin{pmatrix}
X(0) \\
X(4) \\
X(2) \\
X(6) \\
X(1) \\
X(5) \\
X(3) \\
X(7)
\end{pmatrix}
= \begin{pmatrix}
W^0 & W^0 & W^0 & W^0 & W^0 & W^0 & W^0 \\
W^0 & W^4 & W^8 & W^{12} & W^{16} & W^{20} & W^{24} & W^{28} \\
W^0 & W^2 & W^4 & W^6 & W^8 & W^{10} & W^{12} & W^{14} \\
W^0 & W^6 & W^{12} & W^{18} & W^{24} & W^{30} & W^{36} & W^{42} \\
W^0 & W^1 & W^2 & W^3 & W^4 & W^5 & W^6 & W^7 \\
W^0 & W^5 & W^{10} & W^{15} & W^{20} & W^{25} & W^{30} & W^{35} \\
W^0 & W^3 & W^6 & W^9 & W^{12} & W^{15} & W^{18} & W^{21} \\
W^0 & W^7 & W^{14} & W^{21} & W^{28} & W^{35} & W^{42} & W^{49}
\end{pmatrix}
\begin{pmatrix}
x(0) \\
x(1) \\
x(2) \\
x(3) \\
x(4) \\
x(5) \\
x(6) \\
x(7)
\end{pmatrix}
\]

\text{eq. 100}

11. FFT (Fast Fourier Transform) algorithm

We can reduce a number of steps for FFT algorithm with applying a rule eq. 101, eq. 102, eq. 103, eq. 104 of periodic property for multiplication of twiddle factor \( W \). The periodic property of twiddle factor \( W \) is noted below.

\[
W_N^k = W_N^{k \pm \text{m} \cdot N}
\text{eq. 101}
\]

\[
W_N^l \cdot W_N^{k-l}
\text{eq. 102}
\]

\[
W_N^N = 1
\text{eq. 103}
\]

\[
W_N^{N/2} = -1
\text{eq. 104}
\]
FFT algorithm with decimation-in-frequency is shown in Fig. 10. The cross point is called Butterfly operation shown in Fig. 11. The butterfly operation is equation of eq. 105, eq. 106, eq. 107 at Fig. 11.

Fig. 10 FFT algorithm with decimation-in-frequency
Should be read by x[n]→g[n], X[k]→G[k] at eq. 93, eq. 94

Fig. 11 Butterfly operation

\[ W_N^k = \exp \left( -j \frac{2\pi k}{N} \right) = \cos \left( \frac{2\pi k}{N} \right) - j \cdot \sin \left( \frac{2\pi k}{N} \right) = \cos \left( \frac{2\pi k}{N} \right) + j \cdot \cos \left( \frac{2\pi (k + N/4)}{N} \right) \]

eq. 105

\[ A = a + b = a_{real} + j \cdot a_{imag} + b_{real} + j \cdot b_{imag} = (a_{real} + b_{real}) + j \cdot (a_{imag} + b_{imag}) \]

eq. 106
12. C-Programming Code for floating calculation FFT with complex signals

The C-Programming Code for floating calculation FFT with complex signals are shown in Fig. 22, Fig. 23. The Flow of FFT programming is shown in Fig. 13, and also the Explanation diagrams for FFT programming code is shown in Fig. 12. The source code uses the equation eq. 108 and eq. 109.

\[ B = (a - b) \times W_N^k = \left( (a_{\text{real}} - b_{\text{real}}) + j \cdot (a_{\text{imag}} - b_{\text{imag}}) \right) \times W_N^k \]

\[ = \left( (a_{\text{real}} - b_{\text{real}}) + j \cdot (a_{\text{imag}} - b_{\text{imag}}) \right) \times \left\{ \cos \left( \frac{2\pi k}{N} \right) + j \cdot \cos \left( \frac{2\pi (k + N/4)}{N} \right) \right\} \]

\[ = \left( a_{\text{real}} - b_{\text{real}} \right) \cdot \cos \left( \frac{2\pi k}{N} \right) - \left( a_{\text{imag}} - b_{\text{imag}} \right) \cdot \cos \left( \frac{2\pi (k + N/4)}{N} \right) \]

\[ + j \times \left( a_{\text{imag}} - b_{\text{imag}} \right) \cdot \cos \left( \frac{2\pi k}{N} \right) + \left( a_{\text{real}} - b_{\text{real}} \right) \cdot \cos \left( \frac{2\pi (k + N/4)}{N} \right) \]

eq. 107
\[ W_{n,\text{FFT}}[k] = \cos\left(\frac{2\pi}{N} k \right) \quad \text{, } k=0 \text{ to } 3N/4-1 \]

eq. 108

\[ a_{\text{real}} + b_{\text{real}} \Rightarrow xR[j] + xR[jnh] \quad a_{\text{imag}} + b_{\text{imag}} \Rightarrow xI[j] + xI[jnh] \]
\[ a_{\text{real}} - b_{\text{real}} \Rightarrow xR[j] - xR[jnh] \quad a_{\text{imag}} - b_{\text{imag}} \Rightarrow xI[j] - xI[jnh] \]

eq. 109

Fig. 13 Flow of FFT programming

Here, we express the variables of eq. 106, eq. 107 with jxC, jxS such as

\[ jxC \Rightarrow k, jxS \Rightarrow k + \frac{N}{4} \quad \text{eq. 110} \]
13. About FFT Frequency Number Index related with actual signal’s frequency

We have several equations as to FFT Frequency Number Index related with actual signal’s frequency such as noted below.

a) Frequency of ADC sampling

\[ f_{\text{sample}} [Hz] = \frac{1}{T} \quad \text{eq.} \cdot 111 \]

b) Period time of ADC sampling

\[ T [sec] \quad \text{eq.} \cdot 112 \]

c) Angular frequency of ADC sampling

\[ \omega_{\text{sample}} [rad] = \frac{2\pi}{T} \quad \text{eq.} \cdot 113 \]

d) ADC sampling interval time

\[ T_p [sec] = NT \quad \text{eq.} \cdot 114 \]

e) Sampling step of Angular frequency

\[ \Omega [rad] = \frac{2\pi}{T_p} = \frac{\omega_{\text{sample}}}{N} = \frac{2\pi}{NT} \quad \text{eq.} \cdot 115 \]

f) Sampling step of Frequency

\[ \Delta f [Hz] = \frac{\Omega}{2\pi} = \frac{1}{NT} = \frac{f_{\text{sample}}}{N} \quad \text{eq.} \cdot 116 \]

Fig. 14 ADC sampling for Doppler time-waveform signal

---

26
For example:

ADC sampling time : $T=9.6\mu s$
Size of FFT : $N=256$
ADC sampling frequency : $f_{\text{sample}}=104.167\text{kHz}$
Sampling interval time : $T_p=256 \times 9.6\mu s=2.4576\text{ms}$
Sampling step of frequency : $\Delta f=1/2.4576\text{ms}=406.9\text{Hz}$
Max available frequency of FFT : $f_{\text{sample}}/2=52.08\text{kHz}$
Min available frequency of FFT : $\Delta f \times 3=1.22\text{kHz}$
Max available velocity of moving object : 1170km/hr
Min available velocity of moving object : 28km/hr

It is convenient to use the following equations.

g) Max available (Doppler) frequency of FFT

$$f_{d\_\text{Max}}[\text{Hz}] = \frac{f_{\text{sample}}}{2} = \frac{1}{2T} \quad \text{eq. - 117}$$
h) Max available velocity of moving object

\[ v_{\text{Max}}[\text{m/s}] = \frac{f_{d_{-}\text{Max}} \cdot c}{2f_0 \cdot \cos \alpha} = \frac{f_{d_{-}\text{Max}} \cdot \lambda}{2 \cdot \cos \alpha} \]

i) Min available (Doppler) frequency of FFT

\[ f_{d_{-}\text{Min}}[\text{Hz}] = 3 \times \Delta f = \frac{3\Omega}{2\pi} = \frac{3}{NT} = \frac{3}{N} \cdot f_{\text{sample}} \quad \text{eq. 119} \]

j) Min available velocity of moving object

\[ v_{\text{Min}}[\text{m/s}] = \frac{f_{d_{-}\text{Min}} \cdot c}{2f_0 \cdot \cos \alpha} = \frac{f_{d_{-}\text{Min}} \cdot \lambda}{2 \cdot \cos \alpha} \]

\[ = \frac{3f_{\text{sample}} \cdot c}{N \cdot 2f_0 \cdot \cos \alpha} = \frac{3f_{\text{sample}} \cdot \lambda}{N \cdot 2 \cdot \cos \alpha} \]

\[ = \frac{3 \cdot c}{NT \cdot 2f_0 \cdot \cos \alpha} = \frac{3 \cdot \lambda}{NT \cdot 2 \cdot \cos \alpha} \quad \text{eq. 120} \]

k) Resolution of velocity of moving object

Differentiating eq. 4 via \( fd \)

\[ \frac{dv}{df_d} = \frac{d}{df_d} \left( \frac{f_d \cdot c}{2f_0 \cdot \cos \alpha} \right) = \frac{c}{2f_0 \cdot \cos \alpha} \quad \text{eq. 121} \]

So, resolution of velocity of moving object \( \Delta v [\text{m/s}] \) is

\[ \Delta v[m/s] = \frac{c}{2f_0 \cdot \cos \alpha} \times \Delta f[\text{Hz}] \quad \text{eq. 122} \]

By the way, we have estimated the approximate time required of FFT by each processors.

<table>
<thead>
<tr>
<th>Result of estimation as to FFT time</th>
<th>CPU processing</th>
<th>Fix/Float</th>
<th>Size</th>
<th>MIPS</th>
<th>FPU</th>
<th>Language</th>
<th>Time</th>
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<tr>
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<tr>
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<td>Yes</td>
<td>ANSI-C</td>
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<td>ANSI-C</td>
<td>6.496 ms</td>
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<td>Yes</td>
<td>ANSI-C</td>
<td>12.992 ms</td>
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<td>FFT size</td>
<td>Tsample [ms]</td>
<td>Tadc [ms]</td>
<td>( \Delta f ) [Hz]</td>
<td>( f_{d_{-}\text{Max}} ) [kHz]</td>
<td>( f_{d_{-}\text{Min}} ) [kHz]</td>
<td>time(FFT) [ms]</td>
<td>Tot time [ms]</td>
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<td>11.616</td>
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Table 1 Result of estimation as to FFT time
14. Method of searching the several peaks of FFT graph

In the case of searching the several peaks of FFT graph, it is convenient to use the second order differential for Fourier $Y(f)$. It is shown the method of searching the several peaks of FFT graph via the second order differential $\frac{\partial^2 Y(f)}{\partial f^2}$.

![Diagram of FFT graph and peak searching method]

Fig. 16 Method of searching the several peaks of FFT graph

Then, it is shown the algorithm flow of searching peak and the method of determination to peak region.

![Algorithm flow diagram]

Region status = start  
Region status = mid  
Region status = end

freq index = k  
Buf k state OK  
Buf k+1 state OK  
Buf k+2 state OK  
Buf k+3 state OK  
Buf k+4 state OK  
Buf k+5 state Err

freq index = k+5

---

Fig. 17 the method of determination to peak region
Fourier Transform
\( Y(f) = \text{FFT}(y(t)) \)

Determination of Noise Level threshold

Calculate the 1st and 2nd order differential for \( Y(f) \) by frequency \( f \) and set status of region

\[
\frac{\partial Y(f)}{\partial f} \quad \frac{\partial^2 Y(f)}{\partial f^2} \quad \frac{\partial^2 Y(f)}{\partial f^2} < 0
\]

Searching the several peaks of FFT

\[
\frac{\partial Y(f)}{\partial f} = 0
\]

Sorting the peak data by peak amplitude

Fig. 18 the algorithm flow of searching several peaks
We have able to develop the vibration sensor via radio wave with applying the Doppler radar. The Metal surface with vibration occurs the time-dependent phase shift \( \Delta \theta(t) \) of reflecting radio wave.

\[
y_{rs}(t) = A_r(t) \cdot \sin(2\pi \cdot f_0 t + \varphi_L + \varphi_0 + \varphi_r)
\]

\[
y_{rs}(t) = A_r(t) \cdot \sin \left( 2\pi \cdot f_0 t - 2\pi \cdot \frac{L}{\lambda} + \varphi_0 + \varphi_r \right)
\]

\[
y_{rs}(t) = A_r(t) \cdot \sin \left( 2\pi \cdot f_0 t - 2\pi \cdot \frac{2 \cdot R(t)}{\lambda} + \varphi_0 + \varphi_r \right)
\]

\[
y_{rs}(t) = A_r(t) \cdot \sin \left( 2\pi \cdot f_0 t - \frac{4\pi}{\lambda} \cdot (R_0 - v \cdot t) + \varphi_0 + \varphi_r \right)
\]

\[
y_{rs}(t) = A_r(t) \cdot \sin \left( 2\pi \cdot f_0 t - \frac{4\pi f_0}{c} (R_0 - v \cdot t) + \varphi_0 + \varphi_r \right)
\]

\[
y_{rs}(t) = A_r(t) \cdot \sin \left( 2\pi (f_0 + f_d) t - \frac{4\pi R_0}{\lambda} + \varphi_0 + \Delta \theta(t) \right)
\]  eq.- 123

We assume no Doppler signal \((f_d=0)\). The wave carrier term \((f_0 \cdot t)\) is very rapidly changed and certainly detected at detector surface. It is important for observing the vibration to focus the displacement \( \Delta \theta(t) \) of radio wave path. Also the term of path length \( R_0 \) is very important, because of having relations for the boundary conditions of radio wave (namely Nodes and Anti-Nodes). So we take time-average for the rapidly changing carrier term \((f_0 \cdot t)\) in the range of less than vibration period time. Then

\[
\langle y_{rs}(t) \rangle = A_r(t) \cdot \sin \left( 2\pi \cdot f_0 t \right) - \frac{4\pi R_0}{\lambda} + \varphi_0 + \Delta \theta(t) \right)
\]  eq.- 124

The amplitude displacement \( \Delta L(t) \) of mechanical vibration wave is

\[
\Delta L(t) = \Delta L_0 \cdot \sin(2\pi \cdot f \cdot t) = \frac{\Delta \theta(t) \cdot \lambda}{2\pi}
\]  eq.- 125

or

\[
\Delta \theta(t) = \frac{2\pi \cdot \Delta L(t)}{\lambda} = \frac{2\pi \cdot \Delta L_0}{\lambda} \cdot \sin(2\pi \cdot f \cdot t)
\]  eq.- 126

At detector, the strength of detected signal is varied in response to the change of path length \( R_0 \).

It is shown the strength of detected signal with parameter \( R_0 \) at detector. At nodes, eq.- 124 is approximated to eq.- 127 with \( \Delta L_0 \ll \lambda \).

\[
\langle y_{rs}(t) \rangle_{Nodes} = A_r(t) \cdot \sin(\Delta \theta(t)) \equiv A_r(t) \cdot \Delta \theta(t) = A_r(t) \cdot \frac{2\pi \cdot \Delta L_0}{\lambda} \cdot \sin(2\pi \cdot f \cdot t)
\]  eq.- 127

\[31\]
At anti-nodes, eq. 124 is approximated to eq. 128 with $\Delta L_0 \ll \lambda$ and with cut-DC components

$$\langle y_{rx}(t) \rangle_{\text{Anti-Nodes}} = A_r(t) \cdot \cos(\Delta \theta(t)) \equiv A_r(t) \cdot \left(1 - \frac{\Delta \theta^2(t)}{2!}\right)$$

$$\Rightarrow -\frac{A_r(t) \cdot \Delta \theta^2(t)}{2!} = -\frac{A_r(t)}{2!} \cdot \left(\frac{2\pi \cdot \Delta L_0}{\lambda}\right)^2 \cdot \sin^2(2\pi \cdot f_v \cdot t)$$

$$= -\frac{A_r(t)}{2!} \cdot \left(\frac{2\pi \cdot \Delta L_0}{\lambda}\right)^2 \cdot \left\{1 - \cos(2\pi \cdot 2f_v \cdot t)\right\}$$

$$\Rightarrow A_r(t) \cdot \left(\frac{2\pi \cdot \Delta L_0}{\lambda}\right)^2 \cdot \frac{\cos(2\pi \cdot 2f_v \cdot t)}{2}$$

eq. 128

Strength of detected signal $\langle yr \rangle$ [A.U.]

![Diagram](image)

**Fig. 19** the strength of detected signal at detector in response to the change of path length $R_0$

The anti-nodes equation eq. 128 express the double frequency $2^*f_v$ effect.
The ratio between a factor of the double frequency $2^{*}f_{v}$ effect and a factor of the true fundamental signal term $f_{v}$, is noted below.

$$\frac{\langle y_{rs}(t) \rangle_{Nodes}}{\langle y_{rs}(t) \rangle_{Anti-Nodes}} = A_{r}(t) \cdot \sin(\Delta \theta(t)) : A_{r}(t) \cdot \cos(\Delta \theta(t))$$

$$= A_{r}(t) \cdot \Delta \theta(t) : \frac{\Delta L_{0}}{2!} = A_{r}(t) \cdot \frac{2\pi \cdot \Delta L_{0}}{\lambda} \cdot \sin(2\pi \cdot f_{v} \cdot t) : A_{r}(t) \cdot \left(\frac{2\pi \cdot \Delta L_{0}}{\lambda}\right)^{2} \cdot \cos(2\pi \cdot 2f_{v} \cdot t)$$

$$\approx 1 : \frac{2\pi \cdot \Delta L_{0}}{2 \times 2 \times \lambda}$$

eq 129

In the case of K-band ($f_{0}=24$GHz), electromagnetic carrier wavelength $\lambda=12.5$mm, and if the amplitude $\Delta L_{0}$ of a mechanical vibration wave is less than 0.4mm, then the ratio of double frequency $2^{*}f_{v}$ term effect is about 10% or less. See the Fig. 20. This result shown the fact that a deviation $\Delta L_{0}$ with $1/4 \times \lambda$ produces 40% effect of double frequency $2^{*}f_{v}$ term relative to fundamental signal term $f_{v}$.

**Fig. 20 Relation between amplitude of vibration wave and double frequency term**
16. A moving average with overlap method

A moving average with overlap method is very simple and small averaging method. It is shown the algorithm in Fig. 21.

Variable $a_i$ is ADC sampling data, $a_0$ is the first data at start conversion, and $a_N$ is the latest data which is noted below in detail mathematical equation.

$$
\langle a_N \rangle = \frac{a_0}{2^N} + \sum_{i=1}^{N} \frac{2^{-i-1} \times a_i}{2^N} \approx \sum_{i=1}^{N} 2^{-i-N-1} \times a_i \equiv \frac{a_N}{2} + \frac{a_{N-1}}{2^2} + \frac{a_{N-2}}{2^3} + \frac{a_{N-3}}{2^4} + \Lambda
$$  

\text{eq. 130}

Always granting, taking no account of the far past data term which

$$
\frac{2^{-i-1} \times a_i}{2^N} << \frac{a_N}{2}
$$  

\text{eq. 131}

The number of past data index that influence the current sampling data are up until $a_{N-4}$ or $a_{N-7}$ to get the effect of far past data term be less than each 10% or 1%.
Appendix A: FFT C-Source Code

voď fth_table(float wn_FFT[], short br_FFT[], int N_FFT)
{
    int i, n_half, ne, jp;
    float arg;

    /* Calculation of twiddle factor */
    arg = 6.283185307f/N_FFT;
    for (i=0; i<((N_FFT*3)>>2); i++) wn_FFT[i] = cos(arg*i);

    /* Calculation of bit reversal table */
    n_half = N_FFT>>1;
    br_FFT[0] = 0;
    for (ne=1; ne<N_FFT; ne=ne<<1)
    {
        for (jp=0; jp<ne; jp++) br_FFT[jp+ne] = br_FFT[jp] + n_half;
        n_half = n_half>>1;
    }
}

Fig. 22 C-Programming Code for generating tables for FFT
void fft(float xR[], float xI[], float wn_FFT[], short br_FFT[], int N_FFT)
{
    float xtmpR, xtmpI;
    int   j, jnh, k, jxC, jxS, ne, n_half, n_half2;

    n_half = N_FFT>>1;
    for (ne=1; ne<N_FFT; ne=ne<<1)
    {
        n_half2 = n_half<<1;
        for (k=0; k<N_FFT; k=k+n_half2)
        {
            jxC = 0;
            jxS = N_FFT>>2;
            for (j=k; j<(k+n_half); j++)
            {
                jnh = j + n_half;
                /* beginning of butterfly operations */
                xtmpR = xR[j];
                xtmpI = xI[j];
                xR[j] = xtmpR + xR[jnh];
                xI[j] = xtmpI + xI[jnh];
                xtmpR = xtmpR - xR[jnh];
                xtmpI = xtmpI - xI[jnh];
                xR[jnh] = xtmpR*wn_FFT[jxC] - xtmpI*wn_FFT[jxS];
                xI[jnh] = xtmpR*wn_FFT[jxS] + xtmpI*wn_FFT[jxC];
                /* end of butterfly operations */
                jxC = jxC + ne;
                jxS = jxS + ne;
            }
        }
        n_half = n_half>>1;
    }
    /* Bit reverse */
    for (j=0; j<N_FFT; j++)
}
if (j<br_FFT[j])
{
    swap(&xR[j], &xR[br_FFT[j]]);
    swap(&xI[j], &xI[br_FFT[j]]);
}

/* used in FFT procedure */
inline void swap(float *a, float *b)
{
    float tmp;
    tmp = *a;
    *a = *b;
    *b = tmp;
}

Fig.: 23 C-Programming Code for executing FFT